PHYSICAL REVIEW E VOLUME 48, NUMBER 3 SEPTEMBER 1993

Synchronization of chaotic orbits: The influence of unstable periodic orbits

Neelima Gupte and R. E. Amritkar
Department of Physics, University of Poona, Pune 411 007, India
(Received 20 April 1993)

A chaotic trajectory can be synchronized with a desired unstable orbit (chaotic, periodic, or fixed point) by using a drive variable for which the response subsystem Lyapunov exponents (SLE's) are negative. Unexpectedly, for the Lorenz and Rössler systems, the SLE's obtained for synchronization with the fixed point showed good agreement with those obtained for chaotic orbits. For the Duffing oscillator, a similar agreement was found between the SLE's of the chaotic orbit and those of the unstable period-six orbit. It is conjectured that the SLE's of the chaotic orbit retain a memory of the periods of the orbit's origin.

PACS number(s): 05.45.+b

The problem of the control of chaotic systems has recently received much attention [1–4]. For a chaotic system, a freely evolving trajectory cannot be reproduced due to the sensitive dependence on initial conditions and our inability to set the initial conditions precisely. The control of chaos in this context consists of forcing the system to a desired trajectory. Such a trajectory may be chaotic or periodic. In the case of chaotic trajectories Pecora and Carroll [1] have succeeded in forcing a desired chaotic trajectory onto a system by using an appropriate drive variable. They show that if the subsystem Lyapunov exponents (SLE's) corresponding to the remaining or response variables are all negative, the system settles down onto the desired chaotic trajectory.

It is also possible to envisage situations where it is desirable to force a chaotic system onto one of its own unstable periodic orbits for reasons of the enhancement of system performance, or of the adaptability of the system to varying performance requirements. Ott, Grebogi, and Yorke have succeeded in forcing a system onto its own unstable periodic orbits by making a set of small time-dependent perturbations on the system parameters in such a way that the desired periodic orbit is stabilized [2,5,6].

It is reasonable to expect that the stabilization of unstable periodic orbits can be achieved by the method of Pecora and Carroll [1], i.e., by driving the system by an appropriate drive variable. We have found that this is indeed the case. Again, as in the chaotic case [1], we find that the only suitable drive variables are those for which the subsystem Lyapunov exponents are all negative.

An interesting quantity in the above context is the length of the transient required for synchronization with the desired orbit. The length of the transient is controlled by the value of the largest SLE for the desired orbit. If the desired orbits are of different types, i.e., fixed point, periodic, or chaotic, the values of the total Lyapunov exponents of the three types of orbit will be quite distinct from each other. There is thus no *a priori* reason to expect any correlation between the SLE's or the transient times for synchronization for the three types of orbit.

However, we observed a very interesting and unexpected phenomenon in the case of the Lorenz [7], Rössler [8], and Duffing [9] systems. For the Lorenz and Rössler systems the lengths of the transients required for settling onto one of the unstable fixed points of the system and onto the chaotic orbits could be predicted from the knowledge of the SLE's corresponding to the fixed point alone. For each system a comparison of the SLE's of the chaotic orbits with the SLE's of the relevant fixed points showed very good agreement (see Table I). This was startling in view of the fact that the total Lyapunov exponents of the fixed point and the chaotic orbit were quite different. The agreement persisted for different values of the system parameters. Additionally, the length of the transient required for settling onto some of the unstable periodic orbits also appeared to be governed by the SLE's of the same fixed point, and these were in agreement with the SLE's of those unstable periodic orbits. For the Duffing oscillator there was no agreement between the SLE's of the fixed points and those of its chaotic orbits. However, we found that there was good agreement between the SLE's of the unstable period-six orbit [10] of the system and those of its chaotic orbits.

This leads us to speculate that the studied chaotic orbits of the Lorenz and Rössler systems have arisen out of the instability of the fixed point and hence we see an agreement between the SLE's of the fixed point and those of the chaotic orbits for these cases, whereas the chaotic orbits of the Duffing system have come out of the instability of the period-six orbit, giving rise to an agreement between the corresponding SLE's. We thus conjecture that the SLE's of a given chaotic orbit retain the memory of the unstable periods that are its origin.

The above ideas will be discussed in the context of an autonomous n-dimensional dynamical system $\dot{u}=f(u,\mu)$, where $u=(u_1,\dots,u_n)$ and $f(u,\mu)=(f_1(u,\mu),\dots,f_n(u,\mu))$ are n-dimensional vectors and μ is a set of parameters such that the system lies in the chaotic regime. The desired unstable orbit may be chaotic or periodic. It can be predetermined or obtained via a coevolving system [1]. We start the procedure of synchronization [3,4] by dividing the variables of the system into two subsystems, a drive subsystem $u_d=(u_1,\ldots,u_m)$ and a response subsystem $u_r=(u_m+1,\ldots,u_n)$ such that $u=(u_d,u_r)$. The dynamics of each subsystem is governed by

$$\dot{u}_d=f_d(u_d,u_r,\mu)\ , \qquad (1)$$

$$\dot{u}_r=f_r(u_d,u_r,\mu)\ . \qquad (2)$$

In order to lock the given system onto a given unstable orbit O, start the evolution of the system with an initial condition $u'=(u_d',u_r')$, which slightly deviates from the desired orbit such that $u_d'=u_d$ but $u_r'=u_r+\delta u_r$. The drive variable u_d' now evolves according to Eq. (1) and the response variable evolves according to the equation

$$\dot{u}_r'=f_r(u_d,u_r',\mu')\ . \qquad (3)$$

Thus the drive variables of the primed system are continuously set to the drive variables of the desired orbit and the response variables are allowed to evolve freely. The system will settle down onto the desired orbit, provided the drive variables are such that the SLE's corresponding to the response system are all negative [1]. The SLE's of the response system are given by the eigenvalues (time averaged) of the $(n-m)\times(n-m)$-dimensional response subsystem Jacobian matrix J_r whose elements are given by

$$(J_r)_{ij}=\frac{\partial f_i(u_d,u_r',\mu)}{\partial u_j'}\ ,\quad i,j=m+1,\ldots,n\ , \qquad (4)$$

where u_d are the values of the drive variables of the desired trajectory. The length of the transient after which the system settles down onto the desired orbit depends on the value of the largest SLE of the response system.

The simplest orbit by which the system can be driven is by one of its own fixed points, say (u_d^*,u_r^*). The evolution of the response system is governed by Eq. (3) with $u_d=u_d^*$. The SLE's of the response system are given by the eigenvalues of Eq. (4), with u_d replaced by u_d^*. If the real parts of the eigenvalues of the above matrix are all negative the system will synchronize with the fixed point and u_r' will tend to u_r^* after an initial transient.

The total Lyapunov exponents for the fixed points and the chaotic orbit are in general quite different. Also the SLE's clearly depend on the nature of the driving trajectory. Hence *a priori* one might expect that the SLE's of the fixed point (and hence the time taken to synchronize with the fixed point) probably do not have any correlation with those of the chaotic trajectories (and the transients to them), nor is any correlation expected for other unstable periodic orbits. However, actual numerical studies of the Lorenz and Rössler systems showed unexpected results.

We start with the analysis of the Lorenz [7] system for the fixed points. The Lorenz equations have three possible fixed points given by $x^*=y^*=z^*=0.0$ (fixed point 1), and $x^*=y^*=\pm\sqrt{b(r-1)}$, $z^*=r-1$ (fixed points 2 and 3). The subsystem Lyapunov exponents for the three fixed points calculated using the Jacobian matrix J_r [Eq. (4)] are listed in Table I for each possible drive variable. It is clear from Table I that the SLE's corresponding to the x and y as the drive variables are negative for all the three fixed points. Hence the system can be driven to any of the three fixed points using x and y as the driven variables. However, one of the SLE's is positive for the fixed point 1 and is zero for fixed points 2 and 3, with z as the drive variable. The system thus cannot be driven to the fixed points by a z drive.

As noted earlier, a quantity of obvious interest in the present context is the transient time required for the system to settle onto the fixed point. This will of course depend on the drive variable. We study the time T required for the system to approach within some small distance ϵ of the fixed point starting from some initial point. If ϵ_0 is the initial distance from the fixed point, then it is clear that $\epsilon=\epsilon_0\exp(\lambda T)$, where λ is the real part of the largest subsystem Lyapunov exponent.

We study the transient required for settling onto the fixed points 2 or 3 of the Lorenz attractor. We plot the average length of the transient as a function of $\log_{10}\epsilon$ for the drive variables x^* and y^* in Fig. 1. The average is taken over 100 initial conditions for the parameter values $\sigma=10$, $b=\frac{8}{3}$, and $r=60$. Using the slopes of the graph and a conversion to natural logarithms it can be easily seen that $\lambda_x^f=-1.83$ for the drive x^* and $\lambda_y^f=-2.85$ for the drive y. These values of the largest SLE are in good agreement with those obtained analytically and listed in Table I.

We now study a completely distinct case, the transient required for a chaotic orbit to synchronize with the desired chaotic orbit of the system starting from a distinct initial condition. Again we plot the average length of the transient as a function of $\log_{10}\epsilon$ in Fig. 1, where the average is again over 100 initial conditions for the same values of parameter as the data for the fixed point above, i.e., $\sigma=10$, $b=\frac{8}{3}$, and $r=60$. The slopes of the two graphs give $\lambda_x^c=-1.80$ for the drive x and $\lambda_y^c=-2.76$

TABLE I. Subsystem Lyapunov exponents for the Lorenz attractor for different drive variables.

Drive	Fixed point 1 $x^*=y^*=z^*=0$	Fixed points 2,3 $x^*=y^*=\pm\sqrt{b(r-1)}$ $z^*=r-1$	SLE's for $\sigma=10.0,\ r=60.0,\ b=\frac{8}{3}$ Fixed points 2,3	 Chaotic orbit [1]
x	$-b,-1$	$\frac{-b-1\pm\sqrt{(b-1)^2-4b(r-1)}}{2}$	$-1.83,-1.83$	$-1.81,-1.86$
y	$-b,-\sigma$	$-b,-\sigma$	$-2.66,-10.0$	$-2.67,-9.99$
z	$\frac{-\sigma-1\pm\sqrt{(\sigma+1)^2+4\sigma(r-1)}}{2}$	$0,-(\sigma+1)$	$0,-11.0$	$0.0108,\ -11.01$

for the drive y. Again the values of the largest SLE's obtained from the graph are in good agreement with the values obtained numerically for the chaotic orbit by Pecora and Carroll [1].

We note an unexpected and interesting fact. The SLE's for settling onto the fixed point are almost the same as the SLE's for settling onto the chaotic orbit. This is clear from a comparison of the plot of the average transient versus $\log_{10}\epsilon$ for the fixed point and for the chaotic orbit, as plotted in Fig. 1. This can also be confirmed by an examination of Table I. We list the SLE's for the fixed points 2 and 3 for the parameter values $\sigma=10$, $r=60$, and $b=\frac{8}{3}$ in column 4 of Table I. We also list the values of the SLE's reported by Pecora and Carroll [1] for the chaotic orbits at the same set of parameter values in column 5 of Table I. The two sets of values are in striking agreement [12]. We have checked this remarkable agreement at several other sets of parameter values and found similar agreement in all the cases. Thus it appears that the SLE's of the above fixed points govern the behavior of the chaotic orbits of the Lorenz system.

We have carried out a similar analysis for the Rössler attractor [8]. The Rössler system of equations has the fixed points $z^*=-y^*$, $x^*=-ay^*$, and $y^*=(-c\pm\sqrt{c^2-4ab}\,)/2a$. The subsystem Lyapunov exponents of this fixed points show that synchronization is only possible for the drive variable y. The SLE's of the fixed point $z^*=-y^*, x^*=-ay^*$, and $y^*=(-c+\sqrt{c^2-4ab}\,)/2a$ were found to match very well with the SLE's obtained from the plot of the transient time versus $\log_{10}\epsilon$. The SLE's for this fixed point with $a=0.2$, $b=0.2$, and $c=9.0$ are (0.2, -8.99), (0.003, -8.99) and (0.1, 0.1) for the drive variables x^*, y^*, and z^*, respectively. The corresponding SLE's obtained numerically for the chaotic orbits by Pecora and Carroll [1] for the same values of the parameter are (0.2, -8.99), (-0.056, -8.81), and (0.1, 0.1), respectively. We see a good agreement between these sets of values. We also obtained the plot of the average transient for the synchronization of chaotic orbits as a function of $\log_{10}\epsilon$, and the value of the largest SLE obtained from this agreed very well with the value of the largest SLE for the fixed point. We have checked this agreement for several values of system parameters and it holds well in all the cases. Thus the SLE's of one of the fixed points appear to govern the behavior of the chaotic orbits for the Rössler case as well.

The above analysis is for the simplest case of a periodic orbit, i.e., a fixed point. We find that the qualitative behavior observed for the fixed point goes over to the periodic orbits. As an example, we consider the x^2y orbit of the Lorenz map. This orbit is stable in the region $r=99.98$ to 100.795 ($\sigma=10$, $b=\frac{8}{3}$) and persists in an unstable form till $r=47.5$ [11]. We find that the length of the average transient time required to settle down on the orbit is almost independent of r. The plot of the average transient time as a function of $\log_{10}\epsilon$, where ϵ is the distance to which the orbit closure is checked, is a straight line. For $\sigma=10$, $b=\frac{8}{3}$, and $r=60$, we obtain the Lyapunov exponent $\lambda_x^p=-1.827$ for the drive variable x, and $\lambda_y^p=-2.77$ for the drive variable y. We note that

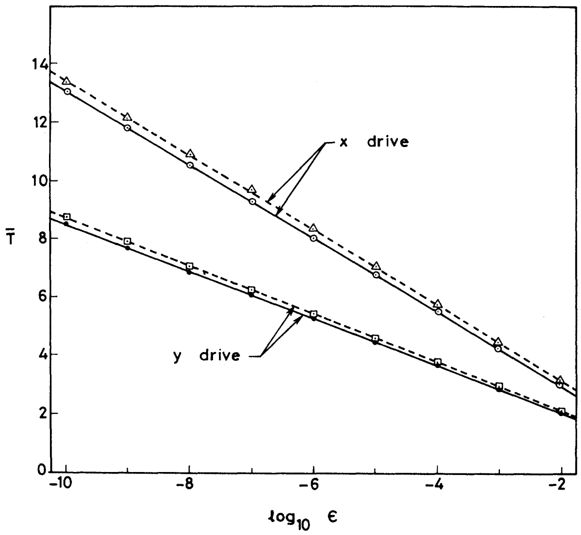

FIG. 1. Plot of average transient time T vs $\log_{10}\epsilon$ for the Lorenz attractor with $\sigma=10$, $b=\frac{8}{3}$, and $r=60$ driven by the unstable fixed point with $x^*=-\sqrt{b(r-1)}$ (hollow circles) and $y^*=-\sqrt{b(r-1)}$ (solid circles) as the drive variables. The triangles and squares show the T values when driven by the chaotic trajectory for the same parameter values and for x and y as the drive variables, respectively.

the Lyapunov exponents for the fixed points 2 and 3 and the periodic orbit agree with each other within our numerical accuracy. Thus for the Lorenz case, the subsystem Lyapunov exponents of the fixed point alone appear to govern the behavior for locking onto the fixed point as well as the chaotic orbits and at least some of the unstable periodic orbits. A similar result was obtained for the unstable periodic orbits of the Rössler attractor.

Next we have investigated the Duffing oscillator [9]; $\dot{x}=y$, $\dot{y}=\frac{1}{2}x(1-x^2)-\gamma y+fz$, $\dot{z}=-\omega u$, and $\dot{u}=\omega z$. This system has a cubic nonlinearity in the equations as opposed to the quadratic nonlinearity of the Lorenz and Rössler systems. We found a distinctly different phenomenon in the case of the Duffing oscillator. (See Table II.) There was no agreement between the SLE's of

TABLE II. Subsystem Lyapunov exponents (real part) for the Duffing oscillator with $\gamma=0.15$, $f=0.17$, $\omega=0.833$, and z as the drive variable are given for the fixed points, period-six orbit, and chaotic orbit. The SLE's for u as the drive variable are the same as z as the drive variable. The SLE's for x and y as the drive variable are (-0.15, 0.0, 0.0) and (0.0, 0.0, 0.0), respectively, for all the cases, i.e., the fixed points, period-six orbit, and chaotic orbit.

Orbit	SLE's
Fixed point 1 $x^*=y^*=z^*=u^*=0$	0.636, 0.0, -0.786
Fixed points 2,3 $x^*=\pm1$, $y^*=z^*=u^*=0$	0.0, -0.075, -0.075
Period-six orbit	0.18, 0.0, -0.33
Chaotic orbit	0.10, 0.0, -0.25

the fixed point of the Duffing system and those of its chaotic orbits. On the other hand, we found that there was reasonable agreement between the SLE's of the unstable period-six orbit [10] of the system and those of its chaotic orbits [13]. This can be seen from Table II, which lists the SLE's for the fixed points, the period-six orbit, and the chaotic orbit for $\gamma = 0.15$, $f = 0.17$, and $\omega = 0.833$. We have checked this result for other values of parameters as well.

We have thus demonstrated that it is possible to stablize the unstable periodic orbits of a chaotic attractor by driving with an appropriate drive variable. The choice of drive variable is dictated by the values of the subsystem Lyapunov exponents of the response system, as is the length of the transient. For the Lorenz and Rössler systems, the SLE's of some of the unstable fixed points appeared to govern the locking to chaotic orbits and the periodic orbits as well. On the other hand, the SLE's of the unstable period-six orbit appear to govern the properties of the chaotic orbit for the Duffing oscillator. It is interesting to note that the actual chaotic trajectories for these systems seem to wind around the unstable fixed points or the periodic orbits for which this agreement between the SLE's is observed.

These results lead us to speculate that the studied chaotic orbits of the Lorenz and Rössler systems have arisen out of the instability of the fixed point [14], and hence we see an agreement between the SLE's of the fixed point and those of the chaotic orbits for these cases, whereas the chaotic orbits of the Duffing system have come out of the instability of the period-six orbit, giving rise to an agreement between the corresponding SLE's. We thus conjecture that the SLE's of a given chaotic orbit retain the memory of the unstable periods that are its origin.

We thank Professor P. K. Kaw and Dr. S. Thomae for useful discussions. We also thank the Department of Science and Technology (India) for financial support in this work.

[1] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[2] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).
[3] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **67**, 945 (1991).
[4] L. M. Pecora and T. L. Carroll, Phys. Rev. A **44**, 2374 (1991).
[5] E. Ott, C. Grebogi, and J. A. Yorke, in *Chaos,* edited by D. K. Campbell (AIP, New York, 1990).
[6] T. Shinbrot, E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **65**, 3215 (1990).
[7] E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).
[8] O. E. Rössler, Phys. Lett. **57**, 397 (1976).
[9] P. J. Holmes, Philos. Trans. R. Soc. London **292**, 419 (1979).
[10] F. C. Moon and G. X. Li, Phys. Rev. Lett. **55**, 1439 (1985).
[11] C. Sparrow, *The Lorenz Equations, Bifurcations and Chaos* (Springer-Verlag, New York, 1982).
[12] Note that the total Lyapunov exponents for the chaotic orbits and the fixed points 2 and 3 are quite different. For the parameter values $\sigma = 10$, $b = \frac{8}{3}$, and $r = 60$, they are (1.37, 0.004, -15.04) and (0.72, 0.72, -15.10), respectively, where only the real parts of the Lyapunov exponents are given.
[13] Unlike the cases of the Lorenz and Rössler attractors, for the Duffing oscillator, the total Lyapunov exponents of the chaotic orbit and the period-six orbit show a reasonable agreement.
[14] For a discussion about the origin of the Lorenz attractor see Ref. [11].